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The Distribution of Share Price Changes

1. INTRODUCTION

This paper presents both theoretical and empirical evidence about a probability distribution which describes the behavior of share price changes. Osborne's Brownian motion theory of share price changes is modified to account for the changing variance of the share market. This produces a scaled *t*-distribution which is an excellent fit to series of share price indices. This distribution is the only known simple distribution to fit changes in share prices. It provides a far better fit to the data than the stable Paretian, compound process, and normal distributions.

2. THE THEORY OF THE DISTRIBUTION OF SHARE PRICE CHANGES

A theory of the distribution of share price changes has been derived by Osborne.[1]

If $p(t)$ represents the price of a share as time t, then $y = \ln[p(t + \tau)/p(t)]$ is the change in the log of price from time t to $t + \tau$. Osborne shows that the prices can be interpreted as an ensemble of decisions in statistical equilibrium, with properties resembling an ensemble of particles in statistical mechanics. The equilibrium distribution of y is given by

$$f(y) = \frac{\exp(-y^2/2\sigma^2\tau)}{\sqrt{2\pi\sigma^2\tau}} \tag{1}$$

where σ^2 is the variance of y over unit time intervals. This distribution is the same as that of a particle in Brownian motion.

The quantity y, the change in the logarithm of price over time interval τ, is the variable studied in the theory of random walks on share prices. This has been studied by a number of authors including Osborne,[2] Moore,[3] Fama,[4] and the author.[5] The theory of Brownian motion used

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1. M. Osborne, "Brownian Motion in the Stock Market," *Operations Research* 7 (1959): 145–73.
2. Ibid.
3. A. B. Moore, "Some Characteristics of Changes in Common Stock Prices," in *The Random Character of Stock Market Prices*, ed. P. Cootner (Cambridge, Mass.: M.I.T. Press, 1964).
4. E. F. Fama, "The Behavior of Stock Market Prices," *Journal of Business* 38 (1965): 34–105.
5. P. Praetz, "Australian Share Prices and the Random Walk Hypothesis," *Australian Journal of Statistics* 11 (1969): 123–39.

by Osborne implies that values of y over nonoverlapping intervals of time constitute a random walk, that is, the sequence of values of y has members which are mutually independent and have a common probability distribution function.

Almost all of the research on the random-walk hypothesis has concentrated on the independence of changes in share prices. However, we are concerned here with the common probability distribution function.

Osborne's work implies that the distribution of share price changes should be normally distributed. However, enough evidence has now accumulated to conclude that the distribution appears to be highly non-normal. Its typical shape, relative to the normal distribution,[6] is of a symmetrical distribution with fat tails, a high, peaked center and hollow in between. Rigorous testing has been carried out by a number of authors including Moore, Fama, and the author. In almost all cases, the distribution of share price changes over small intervals of time (daily or weekly) has been formally rejected as representing a normal distribution. The tests applied have been χ^2 goodness-of-fit tests or tests based on the sample coefficients of skewness or kurtosis. Enough evidence has appeared to reject the normal distribution proposed by Osborne to represent the distribution of share price changes, which followed from his Brownian motion type argument.

One assumption used in the derivation of Osborne's model is that σ^2, the variance of price changes over unit time interval, is a constant. Intuitively, in practice this is not so, because any share market often has long periods of relative activity, followed by long periods of relative inactivity. The information which affects prices does not come uniformly, but rather in bursts of activity. In Brownian motion, σ^2 is proportional to the temperature (constant) of the gas under discussion. Thus, by analogy, we can think of the "temperature" of the share market as being a variable which represents the degree of activity or energy of the markets. Formal evidence has been provided by the author that σ^2 varies significantly from year to year, as the degree of activity in the market also varies.

The extension we wish to include in Osborne's work is that σ^2, the variance of share price changes, is a random variable with distribution function $g(\sigma^2)$. The result obtained by Osborne,

$$f(y) = \frac{\exp(-y^2/2\sigma^2\tau)}{(2\pi\sigma^2\tau)^{\frac{1}{2}}},$$

must be reinterpreted as a conditional distribution—that is, conditional upon a fixed value of σ—which we express as

$$f(y|\sigma^2) = \frac{\exp(-y^2/2\sigma^2\tau)}{(2\pi\sigma^2\tau)^{\frac{1}{2}}}. \qquad (2)$$

6. Fama.

Now we will change equation (2) by considering $\tau=1$ (a unit time interval) and y as having a non-zero mean μ. Neither of these changes affects the generality of the following argument. Thus equation (2) becomes

$$f(y|\sigma^2) = \frac{\exp[-(y-\mu)^2/2\sigma^2]}{(2\pi\sigma^2)^{\frac{1}{2}}}. \tag{3}$$

If we now denote by $h(y)$, the distribution of y which takes into account the random nature of σ^2, we obtain $h(y)$ by

$$h(y) = \int_0^\infty f(y|\sigma^2)\, g(\sigma^2)\, d\sigma^2, \tag{4}$$

with $0 \leqslant \sigma < \infty$.

An acceptable solution, both theoretically and empirically, is given by

$$g(\sigma^2) = \sigma_0^{2m}(m-1)^m\, \sigma^{-2(m+1)}\, e^{-(m-1)\sigma_0^2/\sigma^2}/\Gamma(m)\ . \tag{5}$$

Here, $\sigma_0^2 = E(\sigma^2)$ and the variance of σ^2 is $\sigma_0^4/(m-2)$. This distribution is empirically acceptable, for, as we show later, the distribution for $f(y)$ is an extremely good fit. It is a theoretically acceptable distribution as it is a natural conjugate in the sense of Raiffa and Schlaifer[7] for a sample which is normally distributed. In this case, $g(\sigma^2)$ represents a prior distribution for the unknown parameter σ^2. Thus our approach has an alternative interpretation, that is, the well-known Bayesian method of assigning a prior distribution to an unknown parameter.

When $g(\sigma^2)$ is substituted in equation (4) we obtain $h(y)$ by integration as

$$h(y) = [1 + (y-\mu)^2/\sigma_0^2(2m-2)]^{-m-\frac{1}{2}}\,\Gamma(m)[(2m-2)\pi]^{\frac{1}{2}}\sigma_0. \tag{6}$$

This is a t distribution of $2m = n$ degrees of freedom, except for a scale factor $[n/(n-2)]^{\frac{1}{2}}$.

Thus, the distribution of $(y-\mu)/\sigma_0$ would be that of a scaled t-distribution. When this distribution is plotted against a standard normal distribution for n small, it reproduces the characteristic distribution of share price changes described earlier.

We can also obtain the distribution of $p(\tau)$, the price of a share at time τ, from $y = \ln[p(\tau)/p(0)]$.

Thus, we obtain

$$f[p(\tau)] = \{1 + [\ln p(\tau) - \ln p(0) - \mu\tau]^2/\sigma_0^2\tau(2m-2)\}^{-m-\frac{1}{2}}\Gamma(m+\tfrac{1}{2})/\sigma_0\tau^{\frac{1}{2}}p(\tau)\Gamma(m)[(2m-2)\pi]^{\frac{1}{2}},$$

7. H. Raiffa and R. Schlaifer, *Applied Statistical Decision Theory* (Cambridge, Mass.: Harvard Business School, 1961).

as y has a mean of $\mu\tau$ and variance of $\sigma^2\tau$ over a time interval τ, when $p(0)$ denotes the price of the share at time 0.

The distribution function $g(\sigma^2)$ of the variance has mean σ_0^2, variance $\sigma_0^4/(m-2)$ and a mode at $\sigma_0^2(m-1)/(m+1)$. It is 0 at $\sigma^2 = 0$, rises to a peak and has a long tail to the right. Intuitively, it represents the distribution of the variance (or energy or temperature) of the share under discussion. As such, it represents the changing expectations of investors about interest rates, the state of the economy as a whole, the state of the industry that the company is engaged in, as well as expectations over dividends, earnings, risk, etc., of the particular company. Ideally, we would like to relate all these variables to $g(\sigma^2)$ in a formal analytic manner, but that is a very difficult problem.

The distribution $g(\sigma^2)$ is discussed by Raiffa and Schlaifer as an inverted gamma distribution. It is theoretically possible that other prior distribution exist to describe σ^2, but it would seem unlikely that they would provide a better fit to the data studied. The above approach can be extended to provide a prior distribution for the mean μ by treating it as an unknown parameter like σ^2. The evidence by the author[8] on the variability of yearly means of the data was not nearly as strong as that on the variability of yearly variances. For no series could we regard the variance as being the same from year to year, whereas for more than half the series we could regard the means between years as the same. However, if we do include a normal-inverted gamma joint prior distribution for μ and σ^2, we still obtain a scaled t-distribution of the form of equation (6).

3. THE EMPIRICAL EVIDENCE

The data studied consisted of seventeen share-price index series, which were weekly observations from the Sydney Stock Exchange for the nine years 1958–66—a total of 462 observations. I previously concluded from these data that none of the series were normally distributed.[9] This study used χ^2 goodness-of-fit tests plus tests based on the third and fourth sample moments.

The variable studied here was the change in the logarithm of share price. This variable was standardized with respect to the overall mean and standard deviation of the data, and grouped into twenty-six frequency classes. Expected frequencies were derived from the probabilities of the scaled t-distribution. These probabilities must be calculated directly, as tabulated values are not very useful due to the scaling factor. Fortunately, the cumulative distribution function of this t-distribution can be obtained directly by simple integration methods, although the cases n even and odd must be treated separately. This gives a χ^2 test of the scaled t-distribution against the standardized data described above.

8. Praetz.
9. Ibid.

The values of n studied ranged from 3 to 36. The adequacy of t_n as a distribution for share price changes was judged by a χ^2 goodness-of-fit test on 22 degrees of freedom. We have also fitted to the data the normal distribution, the compound events distribution of Press,[10] and the stable Paretian distribution of Mandelbrot using the probabilities of Fama and Roll.[11]

The results of fitting these different distributions are given in table 1 as χ^2 values on 23, 21, and 22 degrees of freedom. Column 1 contains

Table 1

Chi-squared Results from Fitting Alternative Distributions to Sydney Share Price Indices

Index	Scaled t_n Distribution (n) (df = 22)	Normal Distribution (df = 23)	Compound Events Distribution (df = 21)	Stable Paretian Distribution (α) (df = 22)
1	24.1 (4)	48.1	48.6	44.5 (1.81)
2	21.7 (3)	106.8	106.1	102.6 (1.76)
3	25.1 (4)	43.9	44.0	42.9 (1.90)
4	16.9 (3)	77.9	231.0	75.5 (1.83)
5	25.1 (4)	67.8	220.3	64.7 (1.81)
6	35.4 (4)	82.2	81.9	77.2 (1.77)
7	29.9 (5)	46.6	50.8	43.9 (1.85)
8	22.1 (4)	47.4	28.9	46.9 (1.92)
9	28.6 (5)	44.7	44.7	43.9 (1.91)
10	18.0 (4)	51.9	92.3	43.3 (1.72)
11	29.3 (4)	51.0	36.5	50.8 (1.96)
12	32.2 (4)	59.8	59.9	56.3 (1.82)
13	18.5 (7)	25.7	86.4	25.5 (1.96)
14	19.1 (4)	45.1	45.0	44.3 (1.66)
15	22.5 (4)	66.4	66.4	63.8 (1.82)
16	20.0 (4)	62.1	62.0	58.1 (1.79)
17	21.1 (4)	52.0	51.8	51.4 (1.92)

the lowest χ^2 value from fitting the scaled t_n-distribution, and the value of n which achieved this. Column 2 contains the values from fitting the normal distribution. Column 3 contains the values from fitting the compound events model. In column 4, we have the minimum χ^2 values for the stable Paretian distribution. We have also included an estimate of α, the stable Paretian parameter, derived by interpolating the minimum χ^2 value from the Fama and Roll[12] probabilities. This estimate of α in no way effects the χ^2 values. It merely reflects the value of α and of χ^2 where the series of χ^2 values has a minimum, for the values of $\alpha = 1.0, (.1), 1.9, (.05), 2.0$ tabulated by Fama and Roll.

Using a 1% level of significance, the results are almost unanimous as all the indices are well-fitted by the scaled t-distribution, whereas the

10. S. J. Press, "A Compound Events Model for Security Prices," *Journal of Business* 40 (1967): 317–35.

11. E. F. Fama and R. Roll, "Some Properties of Symmetric Stable Distributions," *Journal of the American Statistical Association* 63 (1968): 817–36.

12. Ibid.

other distributions are rejected in all cases except four. These exceptions are index 13, stable Paretian and normal, and the compound events model on indices 8 and 11. However, even in these cases, the scaled t-distribution has a far better fit, so that, in every possible case, it is far better than the alternative distributions considered.

Of the seventeen indices, the best value of n from the t_n-distribution was given by $n = 3$ twice, $n = 4$ eleven times, $n = 5$ three times, and $n = 7$ once. This is interesting as the variance of $g(v)$ if finite only when $n > 4$, although the variance of t_n is finite for $n > 2$ in this context.

The stable Paretian distribution always provides a better fit than the normal distribution, but the improvement is marginal. Values of the exponent α varied from 1.66 to 1.96.

The compound events model was often similar to these two distributions. Larger χ^2 values in this case were caused by an inability to provide suitable estimates of the parameters. This trouble was also evident in the work of Press.[13]

4. IMPLICATIONS OF THE DISTRIBUTION OF SHARE PRICE CHANGES

For share price indices at least, we have a distribution to describe the behavior of changes in the logarithm of price. As far as we know, this is the only simple distribution which has passed significance tests. Thus, we can make explicit probability statements concerning changes in prices. These probabilities are radically larger for large changes than those based on the normal distribution, which we might have regarded as intuitively reasonable.

For individual share prices, the situation is not as hopeful due to the discrete nature of the price changes and, in particular, to the large number of zero price changes that always seem to occur.

Implicitly, this distribution has something to say about the use of the stable Paretian family of distributions. These were introduced by Mandelbrot,[14] who, in his latest work lists fifteen papers on their theory and application. Mandelbrot has ably demonstrated their advantages, but they have three disadvantages. First, they have an infinite variance, which implies that most conventional statistical theory is inapplicable. Second, their distribution functions are unknown, except in several special cases. Third, the estimation methods for their parameters are not very satisfactory as yet.

Typically, the distributions used by Mandelbrot to represent share price changes are intermediate between a Cauchy and a normal distribution. However, the scaled t-distribution, t_n, also lies between the same

13. Press.

14. B. Mandelbrot, "Long-Run Linearity, Locally Gaussian Process, H-Spectra and Infinite Variances," *International Economic Review* 10 (1969): 82–111.

two distributions. When $n = 1$, we have the Cauchy distribution and when n becomes infinite, we have the normal distribution. Thus the t-distribution must be seriously considered as an alternative to the stable Paretian distributions, as it has none of the latter's disadvantages, and also has a well-developed theory.

The development presented above has clearly scope for expansion. In particular, there may be more theory from theoretical physics which is now applicable, or which may become applicable after simple modification. Also, the distribution of variance may be linked formally with the variables which determine the changes in expectations of share prices. Another extension which appears possible is the use of the t-distribution to jointly represent risk and uncertainty. The t-distribution is based on the normal distribution (risk) with the variance distribution (uncertainty).